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Numerical solution of system of nonlinear second-order integro-differential equations

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ABSTRACT

In this article, numerical solution of a system of nonlinear second-order integro-differential equations with boundary conditions of the Fredholm and Volterra types by means of the Sinc-collocation method is considered. The method is effective for approximation in the case of the presence of end-point singularities. Properties of the Sinc-collocation method required for our subsequent development are given and utilized to reduce the computation of boundary value problems to some algebraic equations. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

1. Introduction

In this paper a Sinc-collocation procedure is developed for the numerical solution of system of nonlinear second-order integro-differential equations of the Fredholm type

$$\begin{cases} \sum_{i=0}^{2}(\mu_{1i}(x)u_1^{(i)}(x)+\eta_{1i}(x)u_2^{(i)}(x))+\lambda_1\int_{\Gamma}K_1(x,t)\varphi_1(t,u_1(t),u_2(t))\mathrm{d}t+\Psi_1(x,u_1(x),u_2(x))=f_1(x),\\ \sum_{i=0}^{2}(\mu_{2i}(x)u_1^{(i)}(x)+\eta_{2i}(x)u_2^{(i)}(x))+\lambda_2\int_{\Gamma}K_2(x,t)\varphi_2(t,u_1(t),u_2(t))\mathrm{d}t+\Psi_2(x,u_1(x),u_2(x))=f_2(x),\\ x,t\in\Gamma=[a,b], \end{cases} \tag{1}$$

and Volterra type

$$\begin{cases} \sum_{i=0}^{2}(\mu_{1i}(x)u_1^{(i)}(x)+\eta_{1i}(x)u_2^{(i)}(x))+\lambda_1\int_{a}^{x}K_1(x,t)\varphi_1(t,u_1(t),u_2(t))\mathrm{d}t+\Psi_1(x,u_1(x),u_2(x))=f_1(x),\\ \sum_{i=0}^{2}(\mu_{2i}(x)u_1^{(i)}(x)+\eta_{2i}(x)u_2^{(i)}(x))+\lambda_2\int_{a}^{x}K_2(x,t)\varphi_2(t,u_1(t),u_2(t))\mathrm{d}t+\Psi_2(x,u_1(x),u_2(x))=f_2(x),\\ x,t\in\Gamma=[a,b], \end{cases} \tag{2}$$

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with boundary conditions

$$u_1(a) = u_1(b) = 0, \qquad u_2(a) = u_2(b) = 0 \tag{3}$$

where $K_i(x,t), f_i(x), i = 1, 2, \mu_{ji}(x)$, and $\eta_{ji}(x), j, i = 1, 2$ are analytic functions and also $\lambda_i, i = 1, 2$ are parameters. $\varphi_i(t, u_1(t), u_2(t))$ and $\Psi_i(x, u_1(x), u_2(x)), i = 1, 2$ are nonlinear in $u_i(x), i = 1, 2$.

Boundary value problems of systems of nonlinear integro-differential equations have a strong physical background and many practical applications in scientific fields such as population and polymer rheology [1,2]. Recently Abbasbandy and Taati [3] have introduced the operational Tau method to solve a system of nonlinear Volterra integro-differential equations with a nonlinear differential part. A simple operational approach, using the Adomian decomposition method, has been proposed for the numerical solution of systems of nonlinear Volterra integro-differential equations in [4]. This method leads to a system of linear algebraic equations. The operational approach to the Tau method is used for the numerical solution of a nonlinear Fredholm integro-differential equations system [5].

A great deal of interest has been focused on applications of the Sinc methods [6,7], these are well addressed in [8–19]. These methods have been used to solve a wide variety mathematical problems. The Sinc-collocation procedures for the eigenvalue problems are presented in [8,9]. The Sinc-collocation method for initial value problems using the globally defined Sinc basis functions was proposed by Carlson et al. [10]. The Sinc–Galerkin scheme has been developed to give approximate solutions for the Korteweg–de Vries model equation in [11]. The Sinc–Galerkin method has been used to give approximate solutions of fifth-order boundary value problems [12]. In [13], an algorithm based on the Sinc function was used for the generation of adaptive radial grids used in density functional theory. This algorithm is general and can be applied for the integration over Slater or Gaussian type functions with only minor modifications, and the relative error of the integration is fully controlled by the algorithm within a specified range of exponential parameters and for a given principal quantum number. A block matrix formulation has been presented for the Sinc–Galerkin technique applied to the wind-driven current problem from oceanography in [14]. In [15], we used a Sinc-collocation procedure for numerical solution of linear Volterra integral equations of the second kind and also in [16], we applied the Sinc method for solving a system of linear Fredholm integral equations. Abdella et al. [17] employed the Sinc method for the dynamic elasto-plastic problem. Stenger [18] derived some Fourier series for the zeta function via Sinc method, also in [19], Stenger has proposed a Sinc approximation for the derivative of the function that is uniformly accurate on the whole interval, finite or infinite. In this paper a global approximation for the solution of a system of nonlinear integro-differential equations with nonlinear differential part (1)–(3) using Sinc functions is developed. Our method consists of reducing the solution of (1)–(3) to a set of algebraic equations. The properties of the Sinc function are then utilized to evaluate the unknown coefficients.

The outline of the paper is as follows. First, in Section 2 we review some of the main properties of the Sinc function and the Sinc method that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the Sinc method may be used to replace Eqs. (1)–(3) by explicit systems of nonlinear algebraic equations, which are solved by *Newton's method*. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples.

2. Preliminaries

The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\mathrm{Sinc}(x) = \begin{cases} \dfrac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{4}$$

For any $h > 0$, the translated Sinc functions with evenly spaced nodes are given as

$$S(j,h)(x) = \mathrm{Sinc}\left(\frac{x - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots. \tag{5}$$

The Sinc function for the interpolating points $x_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{6}$$

Let

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} \mathrm{d}t, \tag{7}$$

then define a matrix $I^{(-1)} = [\delta_{kj}^{(-1)}]$ whose (k,j)th entry is given by $\delta_{kj}^{(-1)}$. If f is defined on the real line, then for $h > 0$ the series

$$C(f,h)(z) = \sum_{j=-\infty}^{\infty} f(jh)\mathrm{Sinc}\left(\frac{z - jh}{h}\right), \tag{8}$$

is called the Whittaker cardinal expansion of f whenever this series converges [6,7]. But in practice we need to use some specific numbers of terms in the above series, such as $j = -N, \dots, N$, where N is the number of Sinc grid points. They are based in the infinite strip D_s in the complex plane

$$D_s = \left\{ w = u + \mathrm{i}v : |s| < d \le \frac{\pi}{2} \right\}. \tag{9}$$

To construct an approximation on the interval (a, b), we consider the conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right). \tag{10}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{ z = x + \mathrm{i}y : \left| arg\left(\frac{z-a}{b-z}\right) \right| < d \le \frac{\pi}{2} \right\}. \tag{11}$$

For the Sinc method, the basis functions on the interval (a, b) for $z \in D_E$ are derived from the composite translated Sinc functions,

$$S_j(z) = S(j, h) \circ \phi(z) = \mathrm{Sinc}\left(\frac{\phi(z) - jh}{h}\right). \tag{12}$$

The function

$$z = \phi^{-1}(w) = \frac{a + b\mathrm{e}^{w}}{1 + \mathrm{e}^{w}} \tag{13}$$

is an inverse mapping of $w = \phi(z)$. We define the range of ϕ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \phi^{-1}(u) \in D_E : -\infty < u < \infty\}. \tag{14}$$

The Sinc grid points $z_k \in (a, b)$ in D_E will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + b\mathrm{e}^{kh}}{b + \mathrm{e}^{kh}}, \quad k = \pm 1, \pm 2, \dots. \tag{15}$$

For further explanation of the procedure, the important class of functions is denoted by $L_\alpha(D_E)$. The properties of functions in $L_\alpha(D_E)$ and detailed discussions are given in [6,7]. We recall the following definitions and theorems for our purpose.

Definition 1. Let $L_\alpha(D_E)$ be the set of all analytic functions u, for which there exists a constant, C, such that

$$|u(z)| \le C\frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D_E,\ 0 < \alpha \le 1. \tag{16}$$

Theorem 1. *Let $\frac{u}{\phi'} \in L_\alpha(D_E)$, let N be a positive integer and let h be selected by the formula*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}, \tag{17}$$

then there exist positive constant c_1, independent of N, such that

$$\left| \int_\Gamma u(z)\mathrm{d}z - h\sum_{k=-N}^{N} \frac{u(z_k)}{\phi'(z_k)} \right| \le c_1 \mathrm{e}^{-(\pi \mathrm{d}\alpha N)^{\frac{1}{2}}}. \tag{18}$$

Theorem 2. *Let $\frac{u}{\phi'} \in L_\alpha(D_E)$, $\alpha > 0$, and $d > 0$, let $\delta_{kj}^{(-1)}$ be defined as in (7), and let $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$. Then there exists a constant, c_2, which is independent of N, such that*

$$\left| \int_a^{z_k} u(t)\mathrm{d}t - h\sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(z_j)}{\phi'(z_j)} \right| \le c_2 \mathrm{e}^{-(\pi \mathrm{d}\alpha N)^{\frac{1}{2}}}. \tag{19}$$

The nth derivative $u(x)$ at some points x_k can be approximated using a finite number of terms as

$$u^{(n)}(x_k) \approx h^{-n} \sum_{j=-N}^{N} \delta_{jk}^{(n)} u_k, \tag{20}$$

where

$$\delta_{jk}^{(n)} = h^n \frac{\mathrm{d}^n}{\mathrm{d}\phi^n} S(j,h) \circ \phi(x)|_{x=x_k}. \tag{21}$$

In particular

$$\delta_{jk}^{(0)} = S(j,h) \circ \phi(x)|_{x=x_k} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \tag{22}$$

$$\delta_{jk}^{(1)} = h \frac{\mathrm{d}}{\mathrm{d}\phi} S(j,h) \circ \phi(x)|_{x=x_k} = \begin{cases} 0, & k = j, \\ \dfrac{(-1)^{(k-j)}}{(k-j)}, & k \neq j, \end{cases} \tag{23}$$

$$\delta_{jk}^{(2)} = h^2 S(k,h) \circ \phi(x)|_{x=x_k} = \begin{cases} \dfrac{-\pi^2}{3}, & k = j, \\ \dfrac{-2(-1)^{(k-j)}}{(k-j)^2}, & k \neq j. \end{cases} \tag{24}$$

3. The Sinc-collocation method

3.1. Nonlinear Fredholm integro-differential equations system

Let us consider the nonlinear Fredholm integro-differential equations system (1) with boundary conditions (3). We assume $\left(u_1(x), u_2(x)\right)$ to be the exact solution of the boundary value problem (1) and let $\left(u_1(x), u_2(x)\right) \in L_\alpha(D_E)$. We consider the Whittaker cardinal expansion (8). The series in relation (8) contains an infinite number of terms. If we let N be a positive integer, then functions $u_1(x)$ and $u_2(x)$ defined over the interval $[a, b]$ are approximated by using a finite number of terms in (8) as

$$u_1(x) \approx u_{1_n}(x) = \sum_{l=-N}^{N} u_{1l} S(j,h) \circ \phi(x), \tag{25}$$

$$u_2(x) \approx u_{2_n}(x) = \sum_{l=-N}^{N} u_{2l} S(j,h) \circ \phi(x), \tag{26}$$

where $\phi(x)$ is defined by (10). For convenience, consider

$$p_i(t) = \varphi_i(t, u_1(t), u_2(t)), \qquad q_i(x) = \Psi_i(x, u_1(x), u_2(x)), \quad i = 1, 2. \tag{27}$$

Now, for the third terms on the left-hand sides of system (1), we suppose that $\frac{K_i(x,.)}{\phi'} p_i \in L_\alpha(D_E),\ i = 1, 2$, then by using Theorem 1, we obtain:

$$\int_\Gamma K_i(x,t) p_i(t) \mathrm{d}t \approx h \sum_{l=-N}^{N} \frac{K_i(x,t_l)}{\phi'(t_l)} p_i(t_l) \tag{28}$$

where

$$p_i(t_l) = \varphi_i(t_l, u_{1l}, u_{2l}), \quad i = 1, 2,\ l = -N, \ldots, N, \tag{29}$$

and u_{jl} denotes an approximate value of $u_j(t_l), j = 1, 2$. Having replaced the third terms on the left-hand sides of system (1) with the Eq. (28), we have

$$\begin{cases} \displaystyle\sum_{i=0}^{2} (\mu_{1i}(x) u_1^{(i)}(x) + \eta_{1i}(x) u_2^{(i)}(x)) + \lambda_1 h \sum_{l=-N}^{N} \frac{K_1(x,t_l)}{\phi'(t_l)} p_1(t_l) + q_1(x) = f_1(x), \\ \displaystyle\sum_{i=0}^{2} (\mu_{2i}(x) u_1^{(i)}(x) + \eta_{2i}(x) u_2^{(i)}(x)) + \lambda_2 h \sum_{l=-N}^{N} \frac{K_2(x,t_l)}{\phi'(t_l)} p_2(t_l) + q_2(x) = f_2(x). \end{cases}$$

Having substituted $x = x_k$ for $k = -N, \ldots, N$, that x_k are Sinc grid points,

$$\begin{aligned} x_k &= \psi(kh) = \phi^{-1}(kh) \\ &= \frac{a + b\mathrm{e}^{kh}}{1 + \mathrm{e}^{kh}}, \end{aligned}$$

also by replacing $u_i(x)$ by $u_{i_n}(x)$, $i = 1, 2$ as in (25) and (26) we get

$$\begin{cases} \sum_{i=0}^{2} (\mu_{1i}(x_k) u_{1_n}^{(i)}(x_k) + \eta_{1i}(x) u_{2_n}^{(i)}(x_k)) + \lambda_1 h \sum_{l=-N}^{N} \frac{K_1(x_k, t_l)}{\phi'(t_l)} p_1(t_l) + q_1(x_k) = f_1(x_k), \\ \sum_{i=0}^{2} (\mu_{2i}(x_k) u_{1_n}^{(i)}(x_k) + \eta_{2i}(x_k) u_{2_n}^{(i)}(x_k)) + \lambda_2 h \sum_{l=-N}^{N} \frac{K_2(x_k, t_l)}{\phi'(t_l)} p_2(t_l) + q_2(x_k) = f_2(x_k), \end{cases} \tag{30}$$

where

$$u'_{i_n}(x) = \sum_{l=-N}^{N} u_{il} [S(j,h)o\phi(x)]', \quad i = 1, 2, \tag{31}$$

$$u''_{i_n}(x) = \sum_{l=-N}^{N} u_{il} [S(j,h)o\phi(x)]'', \quad i = 1, 2. \tag{32}$$

By using (20)–(24) we can obtain,

$$[S(j,h)o\phi(x)]'|_{x=x_k} = \phi' \frac{\mathrm{d}}{\mathrm{d}\phi} [S(j,h)o\phi(x)] \Big|_{x=x_k} = \phi'(x_k) h^{-1} \delta_{jk}^{(1)}, \tag{33}$$

$$\begin{aligned} [S(j,h)o\phi(x)]''|_{x=x_k} &= \left[\phi' \frac{\mathrm{d}}{\mathrm{d}\phi} S(j,h)o\phi(x) \right]' \Big|_{x=x_k} = \phi'' \frac{\mathrm{d}}{\mathrm{d}\phi} [S(j,h)o\phi(x)] + \phi' \frac{\mathrm{d}^2}{\mathrm{d}\phi^2} [S(j,h)o\phi] \Big|_{x=x_k} \\ &= \phi''(x_k) h^{-1} \delta_{jk}^{(1)} + [\phi'(x_k)]^2 h^{-2} \delta_{jk}^{(2)}. \end{aligned} \tag{34}$$

Using (33) and (34), we rewrite (31) and (32) as

$$u'_{i_n}(x_k) = \sum_{l=-N}^{N} u_{il} \{\phi'(x_k) h^{-1} \delta_{jk}^{(1)}\}, \tag{35}$$

$$u''_{i_n}(x_k) = \sum_{l=-N}^{N} u_{il} \{\phi''(x_k) h^{-1} \delta_{jk}^{(1)} + [\phi'(x_k)]^2 h^{-2} \delta_{jk}^{(2)}\}. \tag{36}$$

By replacing (25), (26), (35) and (36) in the system (30), we get the collocation result:

$$\begin{aligned} &\sum_{l=-N}^{N} \left[\phi'(x_k)^2 \mu_{12}(x_k) \frac{\delta_{lk}^{(2)}}{h^2} + [\phi''(x_k) + \mu_{11}(x_k)\phi'(x_k)] \frac{\delta_{lk}^{(1)}}{h} + \mu_{10}(x_k) \delta_{1k}^{(0)} \right] u_{1l} \\ &\quad + \sum_{l=-N}^{N} \left[\phi'(x_k)^2 \eta_{12}(x_k) \frac{\delta_{lk}^{(2)}}{h^2} + [\phi''(x_k) + \eta_{11}(x_k)\phi'(x_k)] \frac{\delta_{lk}^{(1)}}{h} + \eta_{10}(x_k) \delta_{1k}^{(0)} \right] u_{2l} \\ &\quad + \lambda_1 h \sum_{l=-N}^{N} \frac{K_1(x_k, t_l)}{\phi'(t_l)} p_1(t_l) + q_1(x_k) = f_1(x_k), \quad k = -N, \ldots, N. \end{aligned} \tag{37}$$

Similarly for the second equation of (30) we have

$$\begin{aligned} &\sum_{l=-N}^{N} \left[\phi'(x_k)^2 \mu_{22}(x_k) \frac{\delta_{lk}^{(2)}}{h^2} + [\phi''(x_k) + \mu_{21}(x_k)\phi'(x_k)] \frac{\delta_{lk}^{(1)}}{h} + \mu_{20}(x_k) \delta_{1k}^{(0)} \right] u_{1l} \\ &\quad + \sum_{l=-N}^{N} \left[\phi'(x_k)^2 \eta_{22}(x_k) \frac{\delta_{lk}^{(2)}}{h^2} + [\phi''(x_k) + \eta_{21}(x_k)\phi'(x_k)] \frac{\delta_{lk}^{(1)}}{h} + \eta_{20}(x_k) \delta_{1k}^{(0)} \right] u_{2l} \\ &\quad + \lambda_2 h \sum_{l=-N}^{N} \frac{K_2(x_k, t_l)}{\phi'(t_l)} p_2(t_l) + q_2(x_k) = f_2(x_k), \quad k = -N, \ldots, N. \end{aligned} \tag{38}$$

First, we multiply the resulting equations by $h^2/[\phi'(x_k)]^2$ and then since $\delta_{lk}^{(0)} = \delta_{kl}^{(0)}, \delta_{lk}^{(1)} = -\delta_{kl}^{(1)}, \delta_{lk}^{(2)} = \delta_{kl}^{(2)}$ and $\frac{\phi''(x_k)}{[\phi'(x_k)]^2} = -(\frac{1}{\phi'(x_k)})'$, we may rewrite the Eqs. (37) and (38) in the forms

$$\sum_{l=-N}^{N}\left[\mu_{12}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\mu_{11}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\mu_{10}(x_k)}{[\phi'(x_k)]^2}\delta_{kl}^{(0)}\right]u_{1l}$$
$$+ \sum_{l=-N}^{N}\left[\eta_{12}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\eta_{11}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\eta_{10}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{2l}$$
$$+ \lambda_1 h^3\sum_{l=-N}^{N}\frac{K_1(x_k,t_l)}{\phi'(t_l)[\phi(x_k)']^2}p_1(t_l) + \frac{h^2q_1(x_k)}{[\phi'(x_k)]^2} = \frac{h^2f_1(x_k)}{[\phi'(x_k)]^2}, \quad k=-N,\dots,N, \tag{39}$$

and

$$\sum_{l=-N}^{N}\left[\mu_{22}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\mu_{21}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\mu_{20}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{1l}$$
$$+ \sum_{l=-N}^{N}\left[\eta_{22}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\eta_{21}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\eta_{20}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{2l}$$
$$+ \lambda_2 h^3\sum_{l=-N}^{N}\frac{K_2(x_k,t_l)}{\phi'(t_l)[\phi'(x_k)]^2}p_2(t_l) + \frac{h^2q_2(x_k)}{[\phi'(x_k)]^2} = \frac{h^2f_2(x_k)}{[\phi'(x_k)]^2}, \quad k=-N,\dots,N. \tag{40}$$

We set $I^{(m)} = [\delta_{kl}^{(m)}]$, $m = 0, 1, 2$, where $\delta_{kl}^{(m)}$ denotes the (k, l)th element of the matrix $I^{(m)}$. Also, we denote $\mathbf{K}_i = [\lambda_i h^3 \frac{K_i(x_k,t_j)}{[\phi'(x_k)]^2\phi'(t_j)}]$, $i = 1, 2$, $k, j = -N, \dots, N$, and $D(1/\phi') = diag(1/\phi'(x_{-N}), \dots, 1/\phi'(x_N))$. $\mathbf{K}_i$, $i = 1, 2$, and $I^{(m)}$, $m = 0, 1, 2$ are square matrices of order $(2N + 1) \times (2N + 1)$. We then rewrite the Eqs. (39) and (40) in the matrix form which are the nonlinear system

$$\mathbf{A}U + \widetilde{K}.\mathbf{P} + \mathbf{Q} = \mathbf{F}, \tag{41}$$

where

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$
$$A_{11} = D(\mu_{1,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\mu_{1,1}}{\phi'}\right]I^{(1)} + h^2D\left(\frac{\mu_{1,0}}{\phi'^2}\right)I^{(0)},$$
$$A_{12} = D(\eta_{1,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\eta_{1,2}}{\phi'}\right]I^{(1)} + h^2D\left(\frac{\eta_{1,0}}{\phi'^2}\right)I^{(0)},$$
$$A_{21} = D(\mu_{2,1})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\mu_{2,1}}{\phi'}\right]I^{(1)} + h^2D\left(\frac{\mu_{2,0}}{\phi'^2}\right)I^{(0)},$$
$$A_{22} = D(\eta_{2,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\eta_{2,2}}{\phi'}\right]I^{(1)} + h^2D\left(\frac{\eta_{2,0}}{\phi'^2}\right)I^{(0)},$$
$$\widetilde{K} = [\mathbf{K}_1, \mathbf{K}_2]^T, \qquad \mathbf{P} = [p_1(t_l), p_2(t_l)]^T, \quad l = -N, \dots, N,$$
$$\mathbf{Q} = [q_1(x_k), q_2(x_k)]^T, \qquad \mathbf{F} = [f_1(x_k), f_2(x_k)]^T, \quad k = -N, \dots, N,$$
$$U = [u_{1l}, u_{2l}]^T, \quad l = -N, \dots, N.$$

The above nonlinear system consists of $4N + 2$ equations with $4N + 2$ unknowns (u_{1l}, u_{2l}), $l = -N, \dots, N$. Solving this nonlinear system by *Newton's method*, we can obtain an approximation to the solution of system (1):

$$u_{1_n}(x) = \sum_{l=-N}^{N} u_{1l}S(j,h)\circ\phi(x), \qquad u_{2_n}(x) = \sum_{l=-N}^{N} u_{2l}S(j,h)\circ\phi(x).$$

3.2. Nonlinear Volterra integro-differential equations system

Let us consider the system of nonlinear second-order Volterra integro-differential equations (2) with boundary conditions (3). By replacing u_{i_n}, $i = 1, 2$ as in (25) and (26) in the system (2), and then setting $x = x_k$, we get

$$\begin{cases} \displaystyle\sum_{i=0}^{2}(\mu_{1i}(x_k)u_{1_n}^{(i)}(x_k) + \eta_{1i}(x_k)u_{2_n}^{(i)}(x_k)) + \lambda_1 \int_0^{x_k} K_1(x_k,t)\varphi_1(t, u_{1_n}(t), u_{2_n}(t))\mathrm{d}t \\ \qquad\qquad + \Psi_1(x_k, u_{1_n}(x_k), u_{2_n}(x_k)) = f_1(x_k), \\ \displaystyle\sum_{i=0}^{2}(\mu_{2i}(x_k)u_{1_n}^{(i)}(x_k) + \eta_{2i}(x_k)u_{2_n}^{(i)}(x_k)) + \lambda_2 \int_0^{x_k} K_2(x_k,t)\varphi_2(t, u_{1_n}(t), u_{2_n}(t))\mathrm{d}t \\ \qquad\qquad + \Psi_2(x_k, u_{1_n}(x_k), u_{2_n}(x_k)) = f_2(x_k). \end{cases} \tag{42}$$

We consider the notation p_i and q_i as follows:

$$p_i(t) = \varphi_i(t, u_1(t), u_2(t)), \qquad q_i(x) = \Psi_i(x, u_1(x), u_2(x)), \quad i = 1, 2. \tag{43}$$

For the third terms on the left-hand sides of (2), we suppose that $\frac{K_i(x,.)}{\phi'}p_i \in L_\alpha(D_E)$, $i = 1, 2$, then by using Theorem 2, we obtain:

$$\int_a^{x_k} K_i(x_k,t)p_i(t) \approx h\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K_i(x_k,t_l)}{\phi'(t_l)}p_i(t_l), \quad i = 1, 2. \tag{44}$$

Having replaced the third terms on the right-hand sides of (42) with the Eq. (44), using relations (31)–(36), and then multiplying the resulting equations by $h^2/[\phi'(x_k)]^2$, we get the collocation result as

$$\begin{aligned} &\sum_{l=-N}^{N}\left[\mu_{12}(x_k)\delta_{lk}^{(2)} + h\left(\frac{\phi''(x_k)}{[\phi'(x_k)]^2} + \frac{\mu_{11}(x_k)}{\phi'(x_k)}\right)\delta_{lk}^{(1)} + \frac{h^2\mu_{10}(x_k)}{[\phi'(x_k)]^2}\delta_{lk}^{(0)}\right]u_{1l} \\ &\quad + \sum_{l=-N}^{N}\left[\eta_{12}(x_k)\delta_{lk}^{(2)} + h\left(\frac{\phi''(x_k)}{[\phi'(x_k)]^2} + \frac{\eta_{11}(x_k)}{\phi'(x_k)}\right)\delta_{lk}^{(1)} + \frac{h^2\eta_{10}(x_k)}{[\phi'(x_k)]^2}\delta_{lk}^{(0)}\right]u_{2l} \\ &\quad + \lambda_1 h^3\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K_1(x_k,t_l)}{\phi'(t_l)}p_1(t_l) + \frac{h^2q_1(x_k)}{[\phi'(x_k)]^2} = \frac{h^2f_1(x_k)}{[\phi'(x_k)]^2}, \quad k = -N,\ldots,N. \end{aligned} \tag{45}$$

Similarly for the second equation of system (42) we get

$$\begin{aligned} &\sum_{l=-N}^{N}\left[\mu_{22}(x_k)\delta_{lk}^{(2)} + h\left(\frac{\phi''(x_k)}{[\phi'(x_k)]^2} + \frac{\mu_{21}(x_k)}{\phi'(x_k)}\right)\delta_{lk}^{(1)} + \frac{h^2\mu_{20}(x_k)}{[\phi'(x_k)]^2}\delta_{lk}^{(0)}\right]u_{1l} \\ &\quad + \sum_{l=-N}^{N}\left[\eta_{22}(x_k)\delta_{lk}^{(2)} + h\left(\frac{\phi''(x_k)}{[\phi'(x_k)]^2} + \frac{\eta_{21}(x_k)}{\phi'(x_k)}\right)\delta_{lk}^{(1)} + \frac{h^2\eta_{20}(x_k)}{[\phi'(x_k)]^2}\delta_{lk}^{(0)}\right]u_{2l} \\ &\quad + \lambda_2 h^3\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K_1(x_k,t_l)}{\phi'(t_l)}p_2(t_l) + \frac{h^2q_2(x_k)}{[\phi'(x_k)]^2} = \frac{h^2f_1(x_k)}{[\phi'(x_k)]^2}, \quad k = -N,\ldots,N. \end{aligned} \tag{46}$$

Since $\delta_{lk}^{(0)} = \delta_{kl}^{(0)}, \delta_{lk}^{(1)} = -\delta_{kl}^{(1)}, \delta_{lk}^{(2)} = \delta_{kl}^{(2)}$ and also $\frac{\phi''(x_k)}{[\phi'(x_k)]^2} = -(\frac{1}{\phi'(x_k)})'$, we rewrite the Eqs. (45) and (46) as follows:

$$\begin{aligned} &\sum_{l=-N}^{N}\left[\mu_{12}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\mu_{11}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\mu_{10}(x_k)}{[\phi'(x_k)]^2}\delta_{kl}^{(0)}\right]u_{1l} \\ &\quad + \sum_{l=-N}^{N}\left[\eta_{12}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\eta_{11}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\eta_{10}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{2l} \\ &\quad + \lambda_1 h^3\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K_1(x_k,t_l)}{\phi'(t_l)[\phi(x_k)']^2}p_1(t_l) + \frac{h^2q_1(x_k)}{[\phi'(x_k)]^2} = \frac{h^2f_1(x_k)}{[\phi'(x_k)]^2}, \quad k = -N,\ldots,N, \end{aligned} \tag{47}$$

and

$$\sum_{l=-N}^{N}\left[\mu_{22}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\mu_{21}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\mu_{20}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{1l}$$

Table 1
Results for Example 1.

x	$N = 15$		$N = 30$	
	E_{u_1}	E_{u_2}	E_{u_1}	E_{u_2}
0.05	1.90199×10^{-5}	1.63208×10^{-5}	8.09006×10^{-8}	7.55890×10^{-8}
0.15	3.84514×10^{-6}	3.22466×10^{-6}	4.11433×10^{-8}	4.46670×10^{-8}
0.25	1.77507×10^{-5}	1.46520×10^{-5}	2.56548×10^{-10}	9.83273×10^{-9}
0.35	5.44611×10^{-6}	1.03059×10^{-5}	5.01937×10^{-8}	5.09562×10^{-8}
0.45	3.98078×10^{-7}	4.84148×10^{-6}	2.64779×10^{-8}	1.32488×10^{-8}
0.55	7.11875×10^{-7}	5.59600×10^{-6}	2.44740×10^{-8}	6.73549×10^{-8}
0.65	6.04498×10^{-6}	3.50413×10^{-6}	5.14609×10^{-8}	5.22783×10^{-8}
0.75	1.79392×10^{-5}	2.38064×10^{-5}	1.52328×10^{-9}	4.86256×10^{-8}
0.85	4.05263×10^{-6}	2.13177×10^{-6}	4.25878×10^{-8}	1.22093×10^{-9}
0.95	1.91011×10^{-5}	2.54382×10^{-5}	8.15021×10^{-8}	4.25156×10^{-8}

$$+\sum_{l=-N}^{N}\left[\eta_{22}(x_k)\delta_{kl}^{(2)} + h\left(\left(\frac{1}{\phi'(x_k)}\right)' - \frac{\eta_{21}(x_k)}{\phi'(x_k)}\right)\delta_{kl}^{(1)} + \frac{h^2\eta_{20}}{[\phi'(x_k)]^2}(x_k)\delta_{kl}^{(0)}\right]u_{2l}$$
$$+\lambda_2 h^3\sum_{l=-N}^{N}\delta_{kl}^{(-1)}\frac{K_2(x_k,t_l)}{\phi'(t_l)[\phi'(x_k)]^2}p_2(t_l) + \frac{h^2 q_2(x_k)}{[\phi'(x_k)]^2} = \frac{h^2 f_2(x_k)}{[\phi'(x_k)]^2}, \quad k=-N,\ldots,N. \tag{48}$$

By using the notations $D(1/\phi') = diag(1/\phi'(x_{-N}), \ldots, 1/\phi'(x_N))$, $\mathbf{K}_i = [\lambda_i h^3 \frac{K_i(x_k,t_j)}{[\phi'(x_k)]^2\phi'(t_j)}]$, $i = 1, 2$, $k, j = -N, \ldots, N$ and $I^{(m)} = [\delta_{kl}^{(m)}]$, $m = -1, 0, 1, 2$, where $\delta_{kl}^{(m)}$ denotes the (k, l)th element of the matrix $I^{(m)}$, then the system of nonlinear equations (47) and (48) for $(4N + 2)$ unknown coefficients u_{jl}, $j = 1, 2$, $l = -N, \ldots, N$ can be expressed in a matrix form

$$\mathbf{B}U + (I^{(-1)} \circ \widetilde{K})\mathbf{P} + \mathbf{Q} = \mathbf{F}, \tag{49}$$

the notation "$\circ$" denotes the Hadamard matrix multiplication and

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$
$$B_{11} = D(\mu_{1,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\mu_{1,1}}{\phi'}\right]I^{(1)} + h^2 D\left(\frac{\mu_{1,0}}{\phi'^2}\right)I^{(0)},$$
$$B_{12} = D(\eta_{1,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\eta_{1,2}}{\phi'}\right]I^{(1)} + h^2 D\left(\frac{\eta_{1,0}}{\phi'^2}\right)I^{(0)},$$
$$B_{21} = D(\mu_{2,1})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\mu_{2,1}}{\phi'}\right]I^{(1)} + h^2 D\left(\frac{\mu_{2,0}}{\phi'^2}\right)I^{(0)},$$
$$B_{22} = D(\eta_{2,2})I^{(2)} + hD\left[\left(\frac{1}{\phi'}\right)' - \frac{\eta_{2,2}}{\phi'}\right]I^{(1)} + h^2 D\left(\frac{\eta_{2,0}}{\phi'^2}\right)I^{(0)},$$
$$\widetilde{K} = [\mathbf{K}_1, \mathbf{K}_2]^T, \qquad \mathbf{P} = [p_1(t_l), p_2(t_l)]^T, \quad l = -N, \ldots, N,$$
$$\mathbf{Q} = [q_1(x_k), q_2(x_k)]^T, \qquad \mathbf{F} = [f_1(x_k), f_2(x_k)]^T, \quad k = -N, \ldots, N,$$
$$U = [u_{1l}, u_{2l}]^T, \quad l = -N, \ldots, N.$$

Solving this nonlinear system by *Newton's method*, we can obtain an approximation to the solution of (2).

4. Numerical examples

In order to illustrate the performance of the Sinc-collocation method in solving boundary value problems for the integro-differential equations system (1)–(3) and the efficiency of the method, the following examples are considered. The examples have been solved by the presented method with different values of N and α, $0 < \alpha \le 1$. All examples are solved for $d = \frac{\pi}{2}$ and $\alpha = \frac{1}{2}$. We use the absolute error, defined as

$$E_{u_i} = |u_i(x) - u_{i_n}(x)|, \quad i = 1, 2. \tag{50}$$

Tables 1–4 exhibit the absolute errors.

Table 2
Results for Example 2.

x	$N = 15$		$N = 30$	
	E_{u_1}	E_{u_2}	E_{u_1}	E_{u_2}
0.05	2.95648×10^{-4}	2.50852×10^{-5}	2.28492×10^{-6}	6.90628×10^{-8}
0.15	1.27095×10^{-4}	4.61424×10^{-6}	9.47348×10^{-7}	2.41798×10^{-8}
0.25	3.06835×10^{-4}	3.05716×10^{-5}	6.07550×10^{-7}	1.83712×10^{-7}
0.35	4.59667×10^{-5}	3.40906×10^{-5}	1.63063×10^{-6}	3.85671×10^{-8}
0.45	7.58652×10^{-5}	2.68387×10^{-5}	1.72363×10^{-6}	4.58744×10^{-8}
0.55	9.19013×10^{-5}	1.25686×10^{-5}	1.93524×10^{-6}	2.14990×10^{-7}
0.65	1.56370×10^{-5}	9.80675×10^{-6}	1.52179×10^{-6}	9.77765×10^{-8}
0.75	2.97642×10^{-4}	2.84087×10^{-5}	3.86336×10^{-7}	1.23378×10^{-8}
0.85	1.18317×10^{-4}	8.44439×10^{-6}	8.16668×10^{-7}	6.80594×10^{-8}
0.95	2.89698×10^{-4}	2.18935×10^{-5}	2.23331×10^{-6}	3.87862×10^{-8}

Table 3
Results for Example 3.

x	$N = 15$		$N = 30$	
	E_{u_1}	E_{u_2}	E_{u_1}	E_{u_2}
0.05	1.90523×10^{-5}	1.87353×10^{-5}	8.43579×10^{-8}	7.68634×10^{-8}
0.15	5.32072×10^{-6}	2.74481×10^{-6}	4.79864×10^{-8}	3.27255×10^{-8}
0.25	1.70102×10^{-5}	1.74014×10^{-5}	1.10008×10^{-8}	9.80486×10^{-9}
0.35	7.00045×10^{-6}	2.80412×10^{-6}	5.56005×10^{-8}	4.05335×10^{-8}
0.45	1.24877×10^{-6}	1.12208×10^{-6}	2.09687×10^{-8}	3.42598×10^{-8}
0.55	1.36852×10^{-6}	2.22133×10^{-6}	3.86803×10^{-8}	1.15319×10^{-8}
0.65	2.35119×10^{-6}	8.75706×10^{-6}	4.80762×10^{-8}	5.06058×10^{-8}
0.75	1.65062×10^{-5}	1.80710×10^{-5}	7.18187×10^{-9}	1.33453×10^{-8}
0.85	3.06480×10^{-6}	5.69992×10^{-6}	3.35550×10^{-8}	5.06616×10^{-8}
0.95	1.83954×10^{-5}	1.95054×10^{-5}	7.74058×10^{-8}	8.54661×10^{-8}

Table 4
Results for Example 4.

x	$N = 15$		$N = 30$	
	E_{u_1}	E_{u_2}	E_{u_1}	E_{u_2}
0.05	1.93064×10^{-5}	3.15687×10^{-5}	8.17886×10^{-8}	1.71222×10^{-7}
0.15	4.23031×10^{-6}	1.86561×10^{-5}	4.14502×10^{-8}	5.20012×10^{-8}
0.25	1.85555×10^{-5}	5.16999×10^{-5}	3.00945×10^{-9}	1.53079×10^{-7}
0.35	5.27108×10^{-6}	3.56602×10^{-5}	5.66834×10^{-8}	1.82626×10^{-7}
0.45	1.19002×10^{-6}	4.02474×10^{-5}	3.81977×10^{-8}	6.42170×10^{-7}
0.55	5.71346×10^{-7}	4.11206×10^{-5}	3.16220×10^{-8}	6.19236×10^{-7}
0.65	6.46811×10^{-6}	4.09413×10^{-5}	6.05974×10^{-8}	1.37882×10^{-7}
0.75	1.89685×10^{-5}	3.46533×10^{-5}	9.63834×10^{-9}	1.53242×10^{-7}
0.85	4.65579×10^{-6}	1.47524×10^{-5}	4.55344×10^{-8}	1.18939×10^{-8}
0.95	1.94753×10^{-5}	1.27530×10^{-5}	8.32363×10^{-8}	9.10621×10^{-8}

Example 1. Consider the following nonlinear system of Fredholm integro-differential equations with the exact solution $\big(u_1(x), u_2(x)\big) = \big(x^2 - 3x + 2, -x^2 + 3x - 2\big)$.

$$\begin{cases} \dfrac{x^2}{1-x} u_1''(x) + \dfrac{x}{2} u_2'(x) + e^{u_2^2(x)} - \displaystyle\int_1^2 \left(\frac{1}{1+u_1(t)} + u_1(t)u_2(t) - u_2(t) \right) dt = f_1(x), \\ u_2''(x) - \dfrac{x}{2-x} u_2'(x) + u_1^2(x) - \displaystyle\int_1^2 (u_2^2(t) + u_1(t)u_2(t) + u_2(t)) dt = f_2(x), \end{cases} \tag{51}$$
$$u_1(1) = u_1(2) = 0, \qquad u_2(1) = u_2(2) = 0,$$

where

$$f_1(x) = \frac{x^3 - x^2/2 + 3x/2}{1-x} + e^{(-x^2+3x-2)^2} + \frac{-3\sqrt{3} + 10\sqrt{\pi}}{15\sqrt{3}},$$
$$f_2(x) = \frac{2x^2 - x - 4}{2-x} + (x^2 - 3x + 2)^2 + \frac{1}{6}.$$

We solve (51) for $N = 15$ and $N = 30$. In this problem the functions $\mu_{1,2}(x)$ and $\eta_{2,1}(x)$ are singular at $x = 1$ and $x = 2$. The absolute errors are tabulated in Table 1.

Example 2. Consider the boundary value problem

$$\begin{cases} -xu_1''(x) + \dfrac{x}{2}u_1'(x) - u_1(x)e^{u_2(x)} + \displaystyle\int_0^1 (x+t)(u_1^2(t) + u_2^2(t))dt = f_1(x), \\ -\dfrac{x}{3}u_1'(x) - xu_2''(x) + u_2'(x) + \mathrm{Sin}(u_1(t)) + \displaystyle\int_0^1 xt(u_1^2(t) - u_2^2(t))dt = f_2(x), \end{cases} \tag{52}$$

with the boundary conditions

$$u_1(0) = u_1(1) = 0, \qquad u_2(0) = u_2(1) = 0.$$

$f_1(x)$ and $f_2(x)$ are chosen such that the exact solution is $\big(u_1(x), u_2(x)\big) = \big(\sin(\pi x), x^2 - x\big)$. The approximate solutions are calculated for $N = 15$, $N = 30$ and the optimal Sinc mesh size $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. Table 2 exhibits the absolute errors.

Example 3. Consider the following system of Volterra integro-differential equations

$$\begin{cases} u_1''(x) + xu_1'(x) - u_2''(x) - \displaystyle\int_0^x e^{t^2-x}(u_1(t) - tu_2(t) + u_1(t)e^{u_2(t)})dt = f_1(x), \\ u_1'(x) + \dfrac{1}{2}u_2''(x) + \dfrac{x^2}{2}u_2'(x) + \dfrac{1}{1+\mathrm{Cos}(u_1(t))} + \displaystyle\int_0^x e^{t^2+x}(u_1(t) - tu_2(t) - u_1(t)e^{u_2(t)})dt = f_2(x), \end{cases} \tag{53}$$

subject to boundary conditions

$$u_1(0) = u_1(1) = 0, \qquad u_2(0) = u_2(0) = 0,$$

where $f_1(x)$ and $f_2(x)$ are chosen such that the exact solution is $\big(u_1(x), u_2(x)\big) = \big(x^2 - x, x - x^2\big)$. We solve the Example 3 for $N = 15$ and $N = 30$. The absolute errors are tabulated in Table 3 for the parameters $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}$, $h = \pi(\frac{1}{N})^{\frac{1}{2}}$.

Example 4. Consider the following nonlinear boundary value problem with the exact solution $\big(u_1(x), u_2(x)\big) = \big(x-x^2, x^3 - x^2\big)$.

$$\begin{cases} u_1''(x) + \dfrac{x}{2}u_2'(x) + u_2^2(x) - \displaystyle\int_0^x ((x-t)u_2(t) + u_1(t)u_2(t))dt = f_1(x), \\ u_2''(x) + u_1^2(x) - \displaystyle\int_0^x ((x-t)u_1(t) - u_2^2(t) + u_1^2(t))dt = f_2(x), \end{cases} \tag{54}$$

subject to boundary conditions

$$u_1(0) = u_1(1) = 0, \qquad u_2(0) = u_2(0) = 0,$$

where

$$f_1(x) = \frac{7}{6}x^6 - \frac{49}{20}x^5 + \frac{4}{3}x^4 + \frac{3}{2}x^3 - x^2 - 2,$$
$$f_2(x) = \frac{x^7}{7} - \frac{x^6}{3} + \frac{19}{12}x^4 - \frac{5}{2}x^3 + x^2 + 6x - 2.$$

The parameters $\alpha = \frac{1}{2}$, $d = \frac{\pi}{2}\big(h = \pi\big(\frac{1}{N}\big)^{\frac{1}{2}}\big)$ are used for the Sinc-collocation method. The absolute errors are tabulated in Table 4 for $N = 15$ and $N = 30$.

5. Conclusion

The Sinc-collocation method is used to solve the system of nonlinear second-order integro-differential equations with boundary conditions of the Fredholm and Volterra types. The numerical examples show that the accuracy improves with an increasing number of Sinc grid points N.

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